Phys 410
Fall 2014
Lecture #25 Summary
25 November, 2014

We continued our discussion of Special Relativity. Einstein made two postulates:

1) If S is an inertial reference frame and if a second frame S' moves with constant velocity relative to S, then S' is also an inertial reference frame.
2) The speed of light (in vacuum) has the same value c in every direction in all inertial reference frames.

We reviewed the Lorentz transformation for the description of the same event from the perspective of two inertial reference frames S and S' moving at speed V relative to each other:

$$x' = \gamma(x - Vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma(t - xV/c^2)$$

It was noted that this transformation has the appearance of a rotation in a 4-dimensional space spanned by the coordinates x_1, x_2, x_3 (the re-named ordinary Cartesian coordinates) and a new coordinate $x_4 = ct$. The Lorentz transformation can be written in "rotational" form as $x'^{(4)} = \overline{\overline{\Lambda}}\, x^{(4)}$, where $x^{(4)} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ is the space-time 4-vector [which can also be written as $x^{(4)} = (\vec{x}, ct)$, for example] and the 'rotation' matrix representing the Lorentz transformation is $\overline{\overline{\Lambda}} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$. This is not the most general Lorentz transformation. It is a special case called a "boost", which corresponds to a pair of reference frames moving relative to each other along one of the coordinate axes (x_1). Note that we use the superscript $x^{(4)}$ to denote 4-vectors and the vector sign ($\vec{x}$) to denote ordinary 3-vectors.

One can define the rapidity as an angle obeying the equation $\tanh(\varphi) = \beta$, where β is the normalized relative velocity between the two reference frames, as always. With this definition, the Lorentz transformation matrix can be written as

$\overline{\overline{\Lambda}} = \begin{pmatrix} \cosh(\varphi) & 0 & 0 & -\sinh(\varphi) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh(\varphi) & 0 & 0 & \cosh(\varphi) \end{pmatrix}$, which bears a strong resemblance to a rotation matrix in 3-space, except for the use of hyperbolic functions (rather than trigonometric functions) and an extra minus sign. One can think of the Lorentz transformation as a rotation of the 4-space coordinate axes that are used to describe a specific physical event. One nice feature of the rapidity arises in velocity addition. Velocities do not simply add, as we know from the last lecture, but must be combined in a rather peculiar way. On the other hand, rapidities do add linearly; if we add two velocities v_1 and v_2 along the x_1 axis to get the new velocity u, we must use the formula $u = \frac{\mathrm{v}_1+\mathrm{v}_2}{1+\mathrm{v}_1\mathrm{v}_2/c^2}$, whereas for rapidity one simply has $u = \tanh(\varphi_1 + \varphi_2)$. In other words, adding two relativistic velocities is like two consecutive rotations through angles φ_1 and φ_2.

A four-vector is any quantity that transforms under a Lorentz transformation the same way that the space-time 4-vector transforms. Other 4-vectors that we will encounter include velocity and momentum, to be defined later. The first postulate of relativity implies that the laws of physics have the same form in all inertial reference frames. In other words there is no inertial frame in which the laws of physics are particularly simple (e.g. having fewer terms) than any other frame. This suggests that the laws of physics should be Lorentz invariant, meaning that they take exactly the same form after Lorentz transformation. In other words, the laws of physics should be formulated in terms of 4-vectors! Our objective now is to formulate relativistic mechanics in terms of 4-vectors, and to make sure that they reduce to classical Newtonian form in the limit $\frac{V}{c} \ll 1$.

The length of a three dimensional vector $(x^2 + y^2 + z^2)$ does not change after the coordinate system describing the vector is rotated (i.e. $x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2$). This is an example of a scalar invariant of the vector (namely $\vec{x} \cdot \vec{x}$). There is also a scalar product of 4-vectors that always has the same value after an arbitrary Lorentz transformation. The scalar length of a 4-vector is defined as $s \equiv x_1^2 + x_2^2 + x_3^2 - x_4^2$. This can also be written as $s = \vec{x} \cdot \vec{x} - (ct)^2$. Note the minus sign in the last term. This is dictated by the form of the Lorentz transformation given above. We showed by direct calculation with the Lorentz transformation that $s' = x_1'^2 + x_2'^2 + x_3'^2 - x_4'^2 = s$, proving that this is the correct definition of a Lorentz-invariant scalar. Note that s is a scalar that can be positive, negative, or zero.

We applied this scalar invariant to describe the spherical wave emanating from an exploding firework. In frame S at rest relative to the shell just before it explodes, the leading edge of the expanding light sphere is given by the equation $r^2 = (ct)^2$, which is nicely described by the invariant $s = r^2 - (ct)^2 = 0$. Now consider a frame S' moving by at a high rate of speed V relative to S. The scalar invariant in this frame has exactly the same value: $s' = s = 0$. In other words it says that $r'^2 - (ct')^2 = 0$, which means that an observer in S' also sees the light expanding in a spherical (i.e. isotropic) manner at the same speed as the observer in S! This counter-intuitive result is clearly in accordance with the second postulate of relativity.

We next considered the light cone associated with a single event. Take the event to be at the origin of the spatial coordinate system and at time $\mathrm{t} = 0$. If this event is the explosion of a firework, we know from the discussion above that the light expands spherically in (x_1, x_2, x_3) space at a uniform rate c. This is described by the scalar invariant $s = x^{(4)} \cdot x^{(4)} = 0$, or $r^2 = (ct)^2$. All other inertial observers will witness the same value for this scalar invariant, $s' = x^{(4)\prime} \cdot x^{(4)\prime} = 0$, meaning that $r'^2 = (ct')^2$, so that they see the same expanding sphere of light. The locus of points <u>on</u> the expanding sphere is called the light cone, and all inertial observers agree that light itself propagates on the light cone. Now consider a second event that lies <u>inside</u> (and ahead in time $x_4 > 0$) the light cone of the first event. In this case the scalar invariant that describes the second event has a negative value $s < 0$. We showed that all inertial observers in other reference frames also agree on the sign of the scalar invariant and therefore agree on the time-ordering of the two events (namely that the second event occurred after the first event). Hence the second event can be causally related to the first event. Events in the upper part of the light cone ($x_4 > 0$) are therefore in the absolute future of the first event, while those in the mirror image light cone ($x_4 < 0$) are in the absolute past of the event in question. Finally we considered instead a second event which is <u>outside</u> the light cone of the first event. In this case of space-like separation, observers in different reference frames do not necessarily agree on the time ordering of the two events. Therefore the second event cannot be causally linked to the first event.